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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2016

SECOND YEAR [BATCH 2014-17]

Date : 20/05/2016 **MATHEMATICS (Honours)**

Time : 11 am – 3 pm **Paper :** IV Full Marks : 100

[Use a separate Answer Book for each group]

Group - A

Answer any five questions from question no. 1 to 8 : [5×7]

1. a) Define a metric ‘d’ on $\mathbb{N}$ such that $(\mathbb{N}, d)$ has no isolated point. Justify your answer. [5]

   b) Let G be a dense open set in $\mathbb{R}$ and $x \in \mathbb{R}$. Prove that there exist $a, b \in G$ such that $x = a - b$. [2]

2. a) Let (X, d) be a separable metric space. Prove that $\text{Card } X \le c$. [4]

   b) Let A be an uncountable subset of $\mathbb{R}$. Show that A has a limit point. [3]

3. a) If (X,d) is a compact metric space then show that $\exists\, a, b \in X$ such that $d(a,b) = \sup\{d(x,y) : x, y \in X\}$. [4]

   b) Let ‘d_1’ and ‘d_2’ be metrics on $\mathbb{R}^2$ defined by $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ respectively. Prove that ‘d_1’ and ‘d_2’ are equivalent. [3]

4. a) Is a countable metric space connected? Justify your answer. [4]

   b) Prove that a complete countable metric space has an isolated point. [3]

5. a) Prove that every metric space is normal. [4]

   b) Give an example of a closed and bounded subset A of a metric space (X,d) such that A is not compact. [3]

6. a) Prove that any bounded subset of $\mathbb{R}$ is totally bounded. [2]

   b) Show that a metric space X is totally bounded iff every sequence in X has a Cauchy subsequence. [5]

7. a) Define a Lebesgue number of an open cover of a metric space. Find an open cover of $\mathbb{Q}$ having no Lebesgue number. [2+2]

   b) Let $f : [0,1] \to [0,1]$ be a continuous map. Prove that f has a fixed point. [3]

8. a) Prove that $\mathbb{R}$ is connected. [4]

   b) Let X be a connected metric space with at least two distinct points. Show that X is uncountable. [3]

Answer any three questions from question no. 9 to 13 : [3×5]

9. If a power series $\sum_{n=0}^{\infty} a_n x^n$ is neither nowhere convergent nor everywhere convergent, then show that there exists a positive real number R such that the series converges absolutely for all x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$. [5]

10. a) Prove that the series $\sum \frac{x}{n + n^2 x^2}$ is uniformly convergent for all real x. [2]

    b) Prove that the series $\sum (-1)^n x^n (1-x)$ converges uniformly on [0,1], but the series $\sum x^n (1-x)$ is not uniformly convergent on [0,1]. [3]

11. a) Let $\{f_n\}$ be a sequence of function on an interval I that converges uniformly on I to a continuous function f. Let $c \in I$ and $\{x_n\}$ be a sequence in I converging to c. Prove that $\lim_{n\to\infty} f_n(x_n) = f(c)$. [3]

b) Prove that the sequence $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}$, $x \in [0,2]$, is not uniformly convergent on [0,2]. [2]

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function. If $f_n(x) = f\left(x + \frac{1}{n}\right)\ \forall\ x \in \mathbb{R},\ \forall n \in \mathbb{N}$; show that $\{f_n\}$ converges uniformly to f on $\mathbb{R}$. [5]

13. a) If $f_n(x) = x + \frac{1}{n}\ \forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$ then show that $\{f_n^2\}$ is not uniformly convergent on $\mathbb{R}$. [2]

b) Prove that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is continuous over $\mathbb{R}$. [3]

Group - B

Answer any three questions from question no. 14 to 18 : [3×10]

14. a) Solve : $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2}$, in terms of known integral. [5]

b) Solve the equation : $\left\{(x+3)D^2 - (2x+7)D + 2\right\} y = (x+3)^2 e^x$; $D \equiv \frac{d}{dx}$; $D^2 \equiv \frac{d^2}{dx^2}$ by factorisation of operators. [5]

15. a) Solve : $\frac{dx}{dt} + 5x + y = e^t$; $\frac{dy}{dt} - x + 3y = e^{2t}$. [5]

b) Apply Charpit's method to find the complete integral of the equation $p^2x + q^2y = z$; $p \equiv \frac{\partial z}{\partial x}$, $q \equiv \frac{\partial z}{\partial y}$. [5]

16. a) Find the general power series solution near x=2 of $\frac{d^2y}{dx^2} + (x-3)\frac{dy}{dx} + y = 0$. [5]

b) Find the eigen values and eigen functions of the eigen value problem $\frac{d^2y}{dx^2} + \lambda y = 0$, $0 \le x \le \pi$, which satisfies the boundary conditions y = 0 at x = 0 and $\frac{dy}{dx} = 0$ at $x = \pi$. [5]

17. a) Solve : $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$. [4]

b) Using convolution theorem find the value of $L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\}$. [3]

c) Form the partial differential equation by eliminating the arbitrary functions f and g from $z = yf(x) + xg(y)$. [3]

18. a) Find f(y) such that the total differential equation $\left\{(yz+z)/x\right\}dx - z\,dy + f(y)dz = 0$ is integrable. Hence solve it. [4]

b) Using Laplace transform technique, solve $\left\{tD^2 + (1-2t)D - 2\right\} y = 0$, $D \equiv \frac{d}{dt}$ given y(0) = 1 and $y'(0) = 2$. [4]

c) State convolution theorem. [2]

Answer <u>any two</u> questions from question no. <u>19 to 21</u> : [2×10]

19. a) Show that the tangent to the curve : $x^3+y^3=3axy$ at a point other than the origin where it meets the parabola $y^2=4ax$ is parallel to y-axis. [5]

b) Find the pedal equation of the parabola : $r=\dfrac{2a}{1+\cos\theta}$. [5]

20. a) If the polar equation of a curve is $r=f(\theta)$, where f is an even function of θ, show that the curvature of the curve at the point $\theta=0$ is $\dfrac{f(0)-f''(0)}{\{f(0)\}^2}$. [4]

b) Find the oblique asymptotes of the curve : $xy^2-y^2-x^3=0$. [2]

c) Find the envelope of the family of ellipses : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where the parameters a, b are connected by ab = c^2 (c = constant). [4]

21. a) Find the coordinates of the centre of gravity of a circular sector of radius 'a', taking the bisector of the angle as the x-axis, the angle of spread of the sector being 2α. [5]

b) If $I_{m,n}=\int_0^{\pi/2}\cos^m x\sin nx\,dx$, show that $I_{m,m}=\dfrac{1}{2^{m+1}}\left[2+\dfrac{2^2}{2}+\dfrac{2^3}{3}+\ldots+\dfrac{2^m}{m}\right]$, m and n being positive integers. [5]

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